

**THE EXPONENTIAL NEGATIVE BINOMIAL
DISTRIBUTION: A CONTINUOUS BRIDGE BETWEEN
UNDER AND OVER DISPERSION ON A LIFETIME
MODELLING STRUCTURE**

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Abstract

In this paper, we propose a new three parameter distribution with decreasing failure rate distribution. The new distribution contains as particular cases, the exponential geometric and the exponential Poisson distributions proposed by Adamidis and Loukas [3] and Kus [11], respectively. Consequently, as an advantage, it allows for under-dispersion and over-dispersion with respect to a Poisson distribution. We derive expressions for the quantile, r -th raw moments of the new distribution, including the mean and variance, the order statistics, the r -th moment of the order statistics, and the Rényi and Shannon entropy measures. Estimation is carried out via maximum likelihood.

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1. Introduction

Recently, it has emerged among researchers interest in the proposition of simple new survival distributions, which are derived from the usual exponential distribution as competitive distributions of practical significance for the analysis of survival data. For instance, we can cite the exponential-geometric proposed by Adamidis and Loukas [3], the generalized exponential distribution proposed by Gupta and Kundu [10], the exponential-Poisson distribution proposed by Kus [11], and a generalization of the exponential-Poisson distribution proposed by Barreto-Souza and Cribari-Neto [7].

In this paper, we introduce a new three parameter distribution, the so-called exponential negative binomial (ENB) distribution. The new distribution contains the exponential geometric and the exponential Poisson distributions proposed by Adamidis and Loukas [3] and Kus [11], respectively, as particular cases. Consequently, as an advantage, the ENB distribution represents a continuous bridge between under-dispersion and over-dispersion with respect to a Poisson distribution that is usually encountered in practice.

The remainder of the paper is outlined as follows. In Section 2, we introduce the new ENB distribution, present its particular cases and derive its survival and failure rate functions. In Section 3, we derive the expressions for the quantile, r -th raw moments of the ENB distribution, including the mean and variance, the order statistics, the r -th moment of the order statistics, and the Rényi and Shannon entropy measures. In Section 4, we discuss maximum likelihood estimation and inference. Some artificial applications, in Section 5, illustrate the usefulness of the ENB distribution for lifetime modelling. Finally, concluding remarks are addressed in Section 6.

2. The ENB Distribution

The ENB model is derived as follows. Firstly, the negative binomial distribution with parameters α and θ , say $NB(\alpha, \theta)$ (Piegorsch [12]; Ahmed and Abouammoh [4]; Saha and Paul [16]), has probability mass function (pmf) given by

$$p_m = P[M = m] = \frac{\Gamma(\alpha^{-1} + m)}{\Gamma(\alpha^{-1})m!} \left(\frac{\alpha\theta}{1 + \alpha\theta} \right)^m (1 + \alpha\theta)^{-1/\alpha}, \quad (1)$$

$m = 0, 1, 2, \dots$, for $\theta > 0$, $\alpha > -1$, and $\alpha > -1/\theta$, so that,

$$E[M] = \theta \quad \text{and} \quad \text{Var}[M] = \theta + \alpha\theta^2. \quad (2)$$

As $\alpha = 1$ and $\alpha \rightarrow 0$, we obtain the geometric and Poisson distributions, respectively. Regarding negative values of α , Piegorsch [12] pointed out that when $\alpha = -1/\kappa$, for κ a positive integer such that $\kappa > \theta$, the negative binomial distribution with parameters θ and $-1/\kappa$ gives the same probabilities as a binomial distribution with parameters κ and θ/κ . Ross and Preece [15] showed that even if $\kappa = -1/\alpha$ ($\alpha > 0$) is not an integer, the negative binomial distribution still furnishes positive values of $P[M = m]$, $m = 0, 1, \dots, \kappa^*$, where κ^* is the largest integer less than κ . Therefore, α can be called a *dispersion parameter* (Saha and Paul [16]). From (2), it follows that if $-1/\theta < \alpha < 0$, there is underdispersion from the Poisson model. On the other hand, if $\alpha > 0$, the counts are over-dispersed.

In the sequel, we give a characterization of the ENB distribution. Let the discrete random variable M in (1) be zero truncated with pmf given by

$$p_m^* = \frac{\Gamma(\alpha^{-1} + m)}{\Gamma(\alpha^{-1})m!} \left(\frac{\alpha\theta}{1 + \alpha\theta} \right)^m (1 + \alpha\theta)^{-1/\alpha} \left(1 - (1 + \alpha\theta)^{-1/\alpha} \right)^{-1}. \quad (3)$$

Let $\{Y_i\}_{i=1}^M$ be a random sample of an exponential distribution with parameter β for given $M = m$, with probability density function (pdf) given by $f(x|m; \beta) = \beta m e^{-\beta m x}$, where M is assumed here to follow (3). We also assume that $Y_j, j = 1, 2, \dots$ are independent of M .

Then, the random variable $X = \min \{Y_i\}_{i=1}^M$ follows a ENB distribution with pdf given by

$$f(x; \xi) = \theta \beta e^{-\beta x} (1 + \alpha \theta (1 - e^{-\beta x}))^{-\frac{\alpha+1}{\alpha}} \left(1 - (1 + \alpha \theta)^{-1/\alpha}\right)^{-1}, \quad (4)$$

where $\xi = (\beta, \theta, \alpha)$. It is noteworthy that the ENB distribution has a derivable physical interpretation. If there are M components in a series system and their lifetimes are independent and identically distributed following an exponential distribution, then the overall system lifetime X has an ENB distribution.

There are two important particular cases of (4). For $\alpha = 1$, it leads to the EG distribution introduced by Adamidis and Loukas [3]. In this case, $p = \theta / (1 + \theta)$. Thus, Equation (4) implies $f_{EG}(x; \beta, p) = \beta(1 - p) e^{-\beta x} (1 - p e^{-\beta x})^{-2}$ for $p \in (0, 1)$. For $\alpha \rightarrow 0$, we obtain, from (4), the exponential-Poisson density (Kus [11]) given by $f_{EP}(x; \beta, \theta) = \frac{\theta \beta}{(1 - e^{-\theta})} \exp\{-\theta - \beta x + \theta e^{-\beta x}\}$. Also, the ENB distribution represents a continuous bridge between under-dispersion ($-1/\theta < \alpha < 0$) and over-dispersion ($\alpha > 0$) in the counts M .

We also can obtain the X pdf as a mixture form, that is,

$$f(x; \xi) = \sum_{m=1}^{\infty} w_m f_E(x; m\beta), \quad (5)$$

where $f_E(x; m\beta)$ is exponential density function with parameter $m\beta$. The coefficients w_m represent zero truncated negative binomial probability given by (3), that is, $w_m = w_m(\alpha, \theta) = p_m^*$.

Equation (5) reveals that the ENB density is an infinite mixture of exponential densities. Hence, the properties of the ENB distributions can be obtained from an exponential distribution. This mixture form is very useful and holds for any parameter values. The ordinary moment and moment generating function (mgf) of the ENB distribution can be determined from the same infinite weighted linear combination of those quantities for exponential distributions.

From (4), the cumulative distribution function of the random variable X with ENB distribution with parameters β , α , and θ , i.e., $X \sim ENB(\beta, \theta, \alpha)$, is given by

$$F(x; \xi) = 1 - \left((1 + \alpha\theta(1 - e^{-\beta x}))^{-1/\alpha} - (1 + \alpha\theta)^{-1/\alpha} \right) \left(1 - (1 + \alpha\theta)^{-1/\alpha} \right)^{-1}. \quad (6)$$

The survivor and hazard functions are given, respectively, by

$$S(x; \xi) = \left((1 + \alpha\theta(1 - e^{-\beta x}))^{-1/\alpha} - (1 + \alpha\theta)^{-1/\alpha} \right) \left(1 - (1 + \alpha\theta)^{-1/\alpha} \right)^{-1}, \quad (7)$$

and

$$h(x; \xi) = \frac{f(x; \xi)}{S(x; \xi)} = \frac{\theta\beta e^{-\beta x} (1 + \alpha\theta(1 - e^{-\beta x}))^{-\frac{\alpha+1}{\alpha}}}{(1 + \alpha\theta(1 - e^{-\beta x}))^{-1/\alpha} - (1 + \alpha\theta)^{-1/\alpha}}. \quad (8)$$

3. Quantiles, Moments and Entropy Measures

The quantile $\gamma(x_\gamma = F^{-1}(\gamma; \xi))$, for $\gamma \in (0, 1)$ of the ENB distribution follows from (6) as given by

$$x_\gamma = F^{-1}(\gamma; \xi) = \beta^{-1} \log \left(\left(1 - \left((1 - \gamma(1 - (1 + \alpha\theta)^{-1/\alpha}))^{-\alpha} - 1 \right) (\alpha\theta)^{-1} \right)^{-1} \right). \quad (9)$$

Clearly, we can simulate an ENB variate X from an uniform random variable U in $(0, 1)$ by $X = F^{-1}(U; \xi)$.

Some of the most important features and characteristics of a distribution can be studied through its moments, such that mean, variance, tending, dispersion skewness, and kurtosis. Following Genç [9], the r -th ordinary moments of X reduces from (5) to

$$\begin{aligned} E(X^r) &= \beta^{-r} \Gamma(r+1) \sum_{m=1}^{\infty} m^{-r} w_m \\ &= \frac{\theta \beta^{-r} \Gamma(r+1)}{(1+\alpha\theta)[(1+\alpha\theta)^{1/\alpha} - 1]} \\ &\quad \times F_{r+2, r+1} \left(\left[1, \dots, 1, \frac{1+\alpha}{\alpha} \right], [2, \dots, 2], \frac{\alpha\theta}{1+\alpha\theta} \right), \end{aligned} \quad (10)$$

where $F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda)$ is the generalized hypergeometric function. This function is also known as *Barnes's extended hypergeometric function*. The definition of $F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda)$ is

$$F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k \prod_{i=1}^p \Gamma(n_i + k) \Gamma^{-1}(n_i)}{\Gamma(k+1) \prod_{i=1}^q \Gamma(d_i + k) \Gamma^{-1}(d_i)},$$

where $\mathbf{n} = [n_1, n_2, \dots, n_p]$, p is the number of operands of \mathbf{n} , $\mathbf{d} = [d_1, d_2, \dots, d_q]$, and q is the number of operands of \mathbf{d} . Generalized hypergeometric function is quickly evaluated and readily available in standard software such as Maple.

Hence, the mean and variance of the ENB distribution are given, respectively, by

$$E[X] = \frac{\theta \beta^{-1}}{(1+\alpha\theta)[(1+\alpha\theta)^{1/\alpha} - 1]} F_{3,2} \left(\left[1, 1, \frac{1+\alpha}{\alpha} \right], [2, 2], \frac{\alpha\theta}{1+\alpha\theta} \right),$$

and

$$\begin{aligned} \text{Var}[X] &= \frac{\theta\beta^{-2}}{(1+\alpha\theta)[(1+\alpha\theta)^{1/\alpha}-1]} \left[{}_2F_{4,3} \left(\left[1, 1, 1, \frac{1+\alpha}{\alpha} \right], \left[2, 2, 2 \right], \frac{\alpha\theta}{1+\alpha\theta} \right) \right. \\ &\quad \left. - \frac{\theta}{(1+\alpha\theta)[(1+\alpha\theta)^{1/\alpha}-1]} F_{3,2}^2 \left(\left[1, 1, \frac{1+\alpha}{\alpha} \right], \left[2, 2 \right], \frac{\alpha\theta}{1+\alpha\theta} \right) \right]. \end{aligned} \quad (11)$$

The moments generating function (mgf) of X , say $\varphi(t; \xi) = E[\exp(tX)]$, is immediately derived from the mgf of the exponential distribution as

$$\varphi(t; \xi) = \sum_{m=1}^{\infty} w_m \left(1 - \frac{t}{m\beta} \right)^{-1} = \frac{E[1 - t(\beta M)^{-1}]}{(1 - (1 + \alpha\theta)^{-1/\alpha})^{-1}}, \quad (12)$$

where $M \sim ENB(\alpha, \beta)$.

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter problems of estimation and hypothesis testing in a variety of ways. Therefore, we now discuss some properties of the order statistics for our distribution. Let X_1, \dots, X_n be iid random variables following $ENB(\beta, \alpha, \theta)$ distribution.

The pdf of the i -th order statistic, say $X_{i:n}$, is given by

$$f_{i:n}(x; \xi) = \frac{1}{B(i, n-i+1)} f(x; \xi) F^{i-1}(x; \xi) (1 - F(x; \xi))^{n-i},$$

for $i = 1, \dots, n$, where $B(\cdot, \cdot)$ is the beta function. Using the binomial expansion in the last equation, $f_{i:n}(x; \xi)$ becomes

$$f_{i:n}(x; \xi) = \sum_{k=0}^{n-i} \frac{(-1)^k}{B(i, n-i+1)} \binom{n-i}{k} f(x; \xi) F^{i+k-1}(x; \xi), \quad (13)$$

where $f(\cdot)$ and $F(\cdot)$ are pdf and cdf given by (4) and (6), respectively. With a simple changing of variable, it is possible to show that the cdf of $X_{i:n}$ denoted by $F_{i:n}$, becomes

$$F_{i:n}(x; \xi) = \sum_{k=i}^n \binom{n}{k} F^k(x; \xi) (1 - F(x; \xi))^{n-k}. \quad (14)$$

Expressions for the r -th moment of the order statistics $X_{1:n}, \dots, X_{n:n}$ with pdf in the form (13) are obtained by using a result due to Barakat and Abdelkader [5], that is,

$$\begin{aligned} E(X_{i:n}^r) &= r \sum_{k=n-i+1}^n (-1)^{k-n+i-1} \binom{k-1}{n-i} \binom{n}{k} \int_0^\infty x^{r-1} S^k(x; \xi) dx \\ &= r \sum_{k=n-i+1}^n \frac{(-1)^{k-n+i-1}}{(1 - (1 + \alpha\theta)^{-1/\alpha})^k} \binom{k-1}{n-i} \binom{n}{k} \\ &\quad \times \int_0^\infty x^{r-1} \left((1 + \alpha\theta(1 - e^{-\beta x}))^{-1/\alpha} - (1 + \alpha\theta)^{-1/\alpha} \right)^k dx, \end{aligned} \quad (15)$$

for $i = 1, \dots, n$, where $S(\cdot)$ is the survival function in (7).

According to Rényi [14], entropy is a measure of the uncertainty associated with a random variable X and it has been used in several statistical situations. For instance, Abraham and Sankaran [2] introduce and study Rényi entropy for residual lifetime distributions, showing that the proposed measure uniquely determines the distribution and presenting characterizations for some lifetime models. Baratpour et al. [6] presents the information properties of record values based on Shannon entropy measure.

In the sequel, we derive the Rényi and Shannon entropy measures. The Rényi entropy is defined as $I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_{\mathfrak{X}} f^\gamma(x) dx \right\}$, where $\gamma > 0$ and $\gamma \neq 1$ (see Rényi [14]). From (4), we obtain

$$\begin{aligned}
\int_0^\infty f^\gamma(x; \xi) dx &= \left(\frac{\beta(1 + \alpha\theta)^{-1/\alpha}}{\Gamma(\alpha^{-1})(1 - (1 + \alpha\theta)^{-1/\alpha})} \right)^\gamma \\
&\times \sum_{m=1}^\infty \left(\frac{m\Gamma(\alpha^{-1} + m)}{m!} \left(\frac{\alpha\theta}{1 + \alpha\theta} \right)^m \right)^\gamma \int_0^\infty e^{-\gamma m\beta} dx \\
&= \left(\frac{\beta(1 + \alpha\theta)^{-1/\alpha}}{\Gamma(\alpha^{-1})(1 - (1 + \alpha\theta)^{-1/\alpha})} \right)^\gamma \\
&\times \sum_{m=1}^\infty \left(\frac{m\Gamma(\alpha^{-1} + m)}{m!} \left(\frac{\alpha\theta}{1 + \alpha\theta} \right)^m \right)^\gamma \frac{1}{\gamma m\beta}. \tag{16}
\end{aligned}$$

Then, we have

$$\begin{aligned}
I_R(\gamma) &= \frac{1}{1 - \gamma} \log \left\{ \frac{1}{\gamma\beta} \left(\frac{\beta(1 + \alpha\theta)^{-1/\alpha}}{\Gamma(\alpha^{-1})(1 - (1 + \alpha\theta)^{-1/\alpha})} \right)^\gamma \right. \\
&\quad \left. \times \sum_{m=1}^\infty \frac{1}{m} \left(\frac{m\Gamma(\alpha^{-1} + m)}{m!} \left(\frac{\alpha\theta}{1 + \alpha\theta} \right)^m \right)^\gamma \right\}. \tag{17}
\end{aligned}$$

The Shannon entropy is defined as $E[-\log(f(X))]$ (Baratpour et al. [6]). This is a special case obtained from $\lim_{\gamma \rightarrow 1} I_R(\gamma)$ (see Rényi [14]). Hence,

$$\begin{aligned}
E[-\log(f(X))] &= -\log(\theta\beta) + \beta E[X] + \log\left(1 - (1 + \alpha\theta)^{-1/\alpha}\right) \\
&\quad + \left(\frac{\alpha + 1}{\alpha} \right) E[\log(1 + \alpha\theta(1 - e^{-\beta x}))]. \tag{18}
\end{aligned}$$

4. Inference

Let $\mathbf{x} = (x_1, \dots, x_n)$ be a random sample of the ENB distribution with unknown parameter vector $\xi = (\beta, \alpha, \theta)$. The log-likelihood $l = l(\xi; \mathbf{x})$ is given by

$$l = n \left(\log(\theta) + \log(\beta) - \log \left(1 - (1 + \alpha\theta)^{-1/\alpha} \right) \right) - \beta \sum_{i=1}^n x_i - \left(\frac{\alpha + 1}{\alpha} \right) \sum_{i=1}^n \log(1 + \alpha\theta(1 - e^{-\beta x_i})). \quad (19)$$

The maximum likelihood estimates (MLEs) are obtained by direct maximization of the log-likelihood function (19). The advantage of this procedure is that, it runs immediately using existing statistical packages such as R (R Development Core Team [13]). We consider the software R through the simulated annealing algorithm (Aarts and Korst [1]) to compute the MLEs. Interested readers can obtain the code by emailing the authors.

For interval estimation and tests of hypothesis on ξ , it is required the 3×3 unit observed information matrix. For one observation, the observed information matrix $\mathbf{K} = \mathbf{K}(\xi)$ is given by

$$\mathbf{K} = \begin{bmatrix} \kappa_{\beta\beta} & \kappa_{\beta\alpha} & \kappa_{\beta\theta} \\ \kappa_{\alpha\beta} & \kappa_{\alpha\alpha} & \kappa_{\alpha\theta} \\ \kappa_{\theta\beta} & \kappa_{\theta\alpha} & \kappa_{\theta\theta} \end{bmatrix},$$

where

$$\begin{aligned} \kappa_{\beta\beta} &= -\frac{\partial^2 l}{\partial \beta^2} = \frac{1}{\beta^2} - (\alpha + 1)T_1(x + \alpha T_1), \\ \kappa_{\beta\alpha} &= -\frac{\partial^2 l}{\partial \beta \alpha} = T_1 \left(1 - \frac{(\alpha + 1)T_1(1 - e^{-\beta x})}{x e^{-\beta x}} \right), \\ \kappa_{\beta\theta} &= -\frac{\partial^2 l}{\partial \beta \theta} = (\alpha + 1)T_1 \left(\frac{1}{\theta} - \frac{\alpha(1 - e^{-\beta x})}{1 + \alpha\theta(1 - e^{-\beta x})} \right), \\ \kappa_{\alpha\alpha} &= -\frac{\partial^2 l}{\partial \alpha^2} = \frac{2}{\alpha^3} \log(1 + \alpha\theta(1 - e^{-\beta x})) \end{aligned} \quad (20)$$

$$\begin{aligned}
& + \frac{T_1(1 - e^{-\beta x})}{\alpha x e^{-\beta x}} \left(\frac{2}{\alpha} - \frac{(\alpha + 1)T_1(1 - e^{-\beta x})}{x e^{-\beta x}} \right) \\
& - T_2 T_3^2 (1 + T_2) - \frac{T_2}{\alpha} \left(\frac{\theta^2}{(1 + \alpha\theta)^2} - 2T_3 \right), \\
\kappa_{\alpha\theta} & = - \frac{\partial^2 l}{\partial \alpha \theta} = \frac{(1 - e^{-\beta x})}{\theta x e^{-\beta x}} T_1 \left(1 - \frac{(\alpha + 1)}{x e^{-\beta x}} (1 - e^{-\beta x}) T_1 \right) \\
& + \frac{1}{(1 + \alpha\theta)} T_2 T_3 (1 + T_2) - \frac{\theta}{(1 + \alpha\theta)^2} T_2, \\
\kappa_{\theta\theta} & = - \frac{\partial^2 l}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{\alpha(\alpha + 1)(1 - e^{-\beta x})^2}{\theta^2 x^2 e^{-2\beta x}} T_1^2 \\
& - \frac{T_2}{(1 + \alpha\theta)^2} (1 + \alpha + T_2),
\end{aligned}$$

with, $T_1 = (\theta x e^{-\beta x}) / (1 + \alpha\theta(1 - e^{-\beta x}))$, $T_2 = ((1 + \alpha\theta)^{-1/\alpha}) / (1 - (1 + \alpha\theta)^{-1/\alpha})$, and $T_3 = (\log(1 + \alpha\theta)) / \alpha^2 - \theta / (\alpha(1 + \alpha\theta))$.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\xi} - \xi)$ is $N_3(\mathbf{0}, K^{-1}(\xi))$. The estimated asymptotic multivariate normal $N_3(\mathbf{0}, n^{-1}K^{-1}(\hat{\xi}))$ distribution of $\hat{\xi}$ can be used to construct approximate confidence intervals for confidence level $1 - \gamma$ for each parameter ξ_r is given by

$$\left(\hat{\xi}_r - z_{\gamma/2} \sqrt{\hat{\kappa}^{\xi_r, \xi_r}}, \hat{\xi}_r + \sqrt{\hat{\kappa}^{\xi_r, \xi_r}} \right),$$

where $\hat{\kappa}^{\xi_r, \xi_r}$ is the r -th diagonal element of $n^{-1}K^{-1}(\xi)$ estimated at $\xi = \hat{\xi}$, for $r = 1, 2, 3$ and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for testing goodness-of-fit of the ENB distribution and for comparing this distribution with some of its special sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models of the ENB distribution. For example, the LR statistic can be used to check whether the fit using the ENB distribution is statistically “superior” to fit using the EG distribution for a given dataset. In any case, considering the partition $(\xi_1^T, \xi_2^T)^T = (\beta, \alpha, \theta)^T$, tests of hypotheses of the type $H_0 : \xi_1 = \xi_1^{(0)}$ versus $H_A : \xi_1 \neq \xi_1^{(0)}$ can be performed via the LR statistic $\Lambda = 2 \left(l(\hat{\xi}_1, \hat{\xi}_2) - l(\xi_1^{(0)}, \tilde{\xi}_2) \right)$, where $(\hat{\xi}_1, \hat{\xi}_2)$ and $(\xi_1^{(0)}, \tilde{\xi}_2)$ are the MLEs of (ξ_1, ξ_2) under H_A and H_0 , respectively. Under null hypothesis H_0 , $\Lambda \xrightarrow{d} \chi_{\#(\xi_1)}^2$ as $n \rightarrow \infty$, where $\#(\xi_1)$ is the dimension of the vector ξ_1 of interest. The LR test rejects H_0 , if $\Lambda > \Lambda_\gamma$ denotes the upper 100 γ % point of the $\chi_{\#(\xi_1)}$ distribution.

5. Application

In this section, we illustrated the usefulness of the ENB modelling on three dataset generated according to different dispersion schemes (under, equi, over). The data sets are composed by 100 observations each, generated from the ENB distribution.

The first dataset, hereafter A_1 , was simulated from an under-dispersion $ENB(\beta = 1, \theta = 0.1, \alpha = -9)$ distribution. The second dataset, A_2 , was simulated from an equi-dispersion $ENB(\beta = 1, \theta = 10, \alpha = 0)$ distribution. The third dataset, A_3 , was simulated from an over-dispersion $ENB(\beta = 1, \theta = 5, \alpha = 10)$. We compare the fitting of the ENB distribution with the EP distribution (Kus [11]) and the EG distribution (Adamidis and Loukas [3]) by considering the LR statistics values. Table 1

gives the parameter estimates, as well as the $-\max l(\cdot)$ for each fitted distribution for the three considered datasets. Except for the data set A_2 , which was generated according to an equi-dispersion structure, the ENB distribution outperforms its particular case distributions, since we reject the null hypothesis in favour of the ENB distribution at the usual significance levels.

Figure 1 presents the fitted density functions of the three distributions superimposed to the histogram and the fitted survival functions superimposed to the empirical survival one.

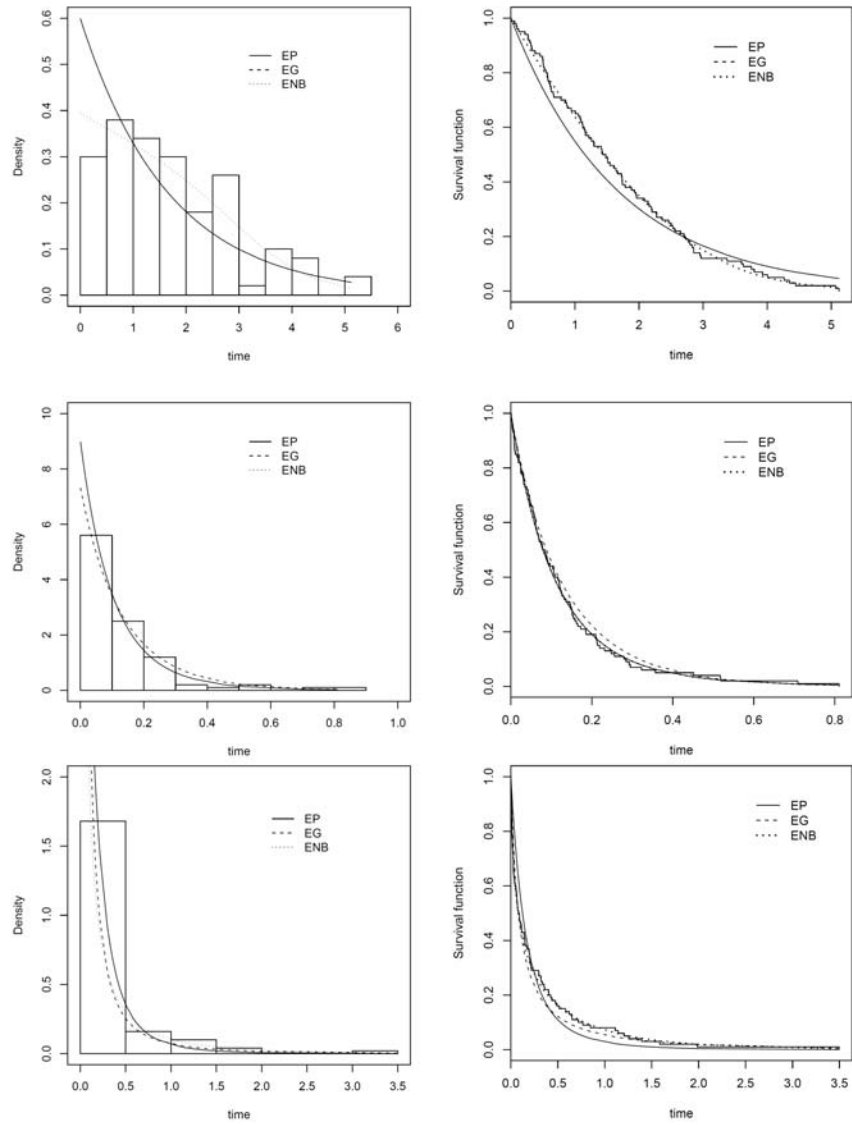


Figure 1. The plots of the fitted ENB, EG, and EP densities. Right panel: Empirical survival function together with some fitted distributions; upper panels: A_1 ; middle panel: A_2 ; bottom panel: A_3 .

Table 1. Parameter estimates and $-\max l(\cdot)$ for the fitted distributions

Data	Distribution	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$-\max l(\cdot)$
A_1 (under-dispersion)	EP	0.60	7.02×10^{-7}	–	145.69
	EG	0.60	6.67×10^{-7}	–	145.69
	ENB	1.37	0.109	–9.05	151.88
A_2 (equi-dispersion)	EP	0.86	10.51	–	109.49
	EG	6.62	0.10	–	108.10
	ENB	0.86	10.51	0.00159	109.49
A_3 (over-dispersion)	EP	1.75	3.37	–	43.65
	EG	0.72	0.94	–	54.46
	ENB	1.19	5.70	10.46	58.28

6. Conclusion

We propose the ENB distribution generalizing the EG and EP distributions, proposed by Adamidis and Loukas [3] and Kus [11], respectively, with the third parameter α bringing flexibility with respect to the EP and EG distributions. We provide a mathematical treatment of the new distribution including expansions for its density, survival, hazard and distribution functions, moments, and quantile function. We obtain the pdf of the order statistics and provide expansions for the moments of the order statistics. We also provide two entropy measures. The estimation of parameters is based on the maximum likelihood approach. We derive the observed information matrix, give asymptotic confidence intervals for the distribution parameters, and consider the use of the LR statistic to compare the fit of the ENB model with its particular cases. We fit the ENB model to three different data sets in order to show the flexibility and the potentially of the new distribution, where we observed a better fit of the ENB distribution compared to its particular cases in the case, when presence of under or over dispersion is observed.

In many practical applications, the lifetimes are affected by explanatory variables. In this context, parametric regression structure for estimating univariate are widely used, since, when the parametric regression models provide a good fit to the lifetime dataset, they tend to give more precise estimates of the quantities of interest because, these estimates are based on fewer parameters. Regression structuring should be investigated further in the context of the proposed ENB distribution.

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References

- [1] E. Aarts and J. Korst, *Simulated Annealing and Boltzmann Machines*, John Wiley and Sons, New York, 1990.
- [2] B. Abraham and P. G. Sankaran, Renyi's entropy for residual lifetime distribution, *Statistical Papers* 47 (2006), 17-29.
- [3] K. Adamidis and S. Loukas, A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters* 39 (1998), 35-42.
- [4] A. N. Ahmed and A. M. Abouammoh, Characterizations of gamma, inverse Gaussian, and negative binomial distributions via their length-biased distributions, *Statistical Papers* 34 (1993), 167-173.
- [5] H. M. Barakat and Y. H. Abdelkader, Computing the moments of order statistics from nonidentical random variables, *Statistical Methods and Applications* 13 (2004), 15-26.
- [6] S. Baratpour, J. Ahmadi and N. R. Arghami, Entropy properties of record statistics, *Statistical Papers* 48 (2007), 197-213.
- [7] W. Barreto-Souza and F. Cribari-Neto, A generalization of the exponential-poisson distribution, *Statistics and Probability Letters* 79 (2009), 2493-2500.
- [8] M. Blagojevic and G. MacKenzie, A Comparison of PH and Non-PH Frailty Models, *Proceedings of the Workshop on Correlated Data Modelling*, Torino, Italy, 9-10 January 2004, pages 27-29.
- [9] A. I. Genç, Moments of order statistics of topp-leone distribution, *Statistical Papers* (2010), DOI 10.1007/s00362-010-0320-y.
- [10] R. Gupta and D. Kundu, Generalized exponential distributions, *Australian and New Zealand Journal of Statistics* 41 (1999), 173-188.

- [11] C. Kus, A new lifetime distribution, *Computational Statistics and Data Analysis* 51 (2007), 4497-4509.
- [12] W. W. Piegorsch, Maximum likelihood estimation for the negative binomial dispersion parameter, *Biometrics* 46 (1990), 863-867.
- [13] R Development Core Team, *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, 2009, ISBN 3-900051-07-0.
- [14] A. Rényi, On measures of entropy and information, *Proceedings of the 4th Berkeley Symposium on Mathematics Statistics and Probability* 1 (1961), 547-561.
- [15] G. Ross and D. Preece, The negative binomial distribution, *The Statistician* 34 (1985), 323-336.
- [16] K. Saha and S. Paul, Bias-corrected maximum likelihood estimator of the negative binomial dispersion parameter, *Biometrics* 61 (2005), 179-185.

